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On Asymptotically Periodic Linear Systems

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1. INTRODUCTION

Among the few cases of explicit solubility of an n-dimensional first-order differential equation

$$y' = A(t)\,y, \qquad t \geqslant t_0, \tag{1.1}$$

are those in which the coefficient-matrix $A(t)$ is constant, or is periodic with real positive period ω. The solution is explicit in the former case in that it depends on the eigenvalues, and reduction to canonical form, of the constant matrix A_0; in the periodic case, it depends on a similar reduction of a fundamental matrix solution, taken over a period, the place of eigenvalues being taken by certain characteristic exponents.

A great deal of work (see e.g., [1-5]) has been devoted to extending the explicit solution for the constant coefficient case to explicit approximate solutions for cases in which the coefficients are in some sense asymptotically constant. The simplest case is that in which there is a constant matrix A_0 such that $\int_{t_0}^{\infty} | A(t) - A_0 |\, dt < \infty$; here we assume the entries in $A(t)$ integrable over any finite interval, and indicate by $| \ |$ any of the usual norms for square matrices. In this case we seek an approximation between the solutions of (1.1) and those of $y' = A_0 y$. An important extension is due to Levinson [2, 3, 5, 6], who has considered the form

$$A(t) = A_0 + B(t) + C(t), \tag{1.2}$$

where A_0 is constant, $B(t)$ is of bounded variation over (t_0, ∞) with $B(\infty) = 0$, and $\int_{t_0}^{\infty} | C(t) |\, dt < \infty$. In this case we have to work in terms of the variable eigenvalues of $A_0 + B(t)$, rather than those of A_0.

The investigation of certain special classes of differential equations shows that the condition that $B(t)$ be of bounded variation can play an essential role. In particular, there are relevant phenomena of this kind in connection with the second-order scalar equation

$$u'' + (1 + f(t))\,u = 0, \qquad t \geqslant t_0, \tag{1.3}$$

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in which $f(t) \to 0$ as $t \to +\infty$, but in which the behavior of the solutions departs in some way or other from the behavior of $u'' + u = 0$. It is a classical observation that the condition $f(t) \to 0$ does not suffice to ensure the boundedness of solutions [7]. Here it is appropriate to cite investigations of (1.3) with $f(t) = h(t) \cos kt$. If $k = 1, 2$ and $h(t)$ tends to zero smoothly but not too rapidly, we get unbounded solutions as $t \to \infty$ [4, 8], whereas the theory of (1.2), when applicable, predicts for (1.3) a pair of solutions asymptotic to

$$\cos \int_{t_0}^{t} (1 + f(s))^{1/2}\, ds, \qquad \sin \int_{t_0}^{t} (1 + f(s))^{1/2}\, ds. \tag{1.4}$$

For other values of k, and suitable conditions on $h(t)$, we get bounded solutions, but with a phase differing from that given by (1.4). This has been discussed in [8, 9], and in a forthcoming paper by Kelman and Madsen [10].

In equations such as (1.3) with $f(t) = h(t) \cos kt$, we have to deal with a perturbation of a constant coefficient system, in which the perturbing term $h(t) \cos kt$ combines smallness at infinity, on account of the function $h(t)$, with periodic variation, on account of the function $\cos kt$. There is a rather obvious analogy with perturbation by periodic coefficients involving small parameters, and indeed some correspondence in respect of the results. Thus, for example, the equation $u'' + (1 + \epsilon \cos kt)\, u = 0$ also has unbounded solutions when $k = 1, 2$ for small nonzero ϵ. The aim of the present paper is to make this connection explicit, at least for the stable case.

We work in terms of a system (1.1) where $A(t)$ is asymptotic to a periodic function, which could of course be constant; the perturbation, or discrepancy between $A(t)$ and this limiting periodic function will of course tend to zero, but will involve also periodically varying ingredients. As an example, we could have

$$A(t) = \sum_{r=-N}^{N} P_r(t) \exp\left(\frac{2\pi i r t}{\omega}\right), \tag{1.5}$$

where $\omega > 0$ is the period, and the $P_r(t)$ are of bounded variation over (t_0, ∞). With such a system we can associate a family of equations with periodic coefficients, namely

$$y' = \left\{\sum_{-N}^{N} P_r(v) \exp\left(\frac{2\pi i r t}{\omega}\right)\right\} y, \tag{1.6}$$

where v is a parameter. The idea is then to consider the characteristic exponents of this system, which are functions of v, in much the same way as one considers the eigenvalues of the variable matrix $A_0 + B(t)$ in the case of (1.2). Indeed, we shall proceed by reducing the problem to the case of (1.2), and appealing to Levinson's result.

2. The Main Result

We use a more general form that (1.5), setting

$$A(t) = B(t, t), \qquad t \geqslant t_0, \tag{2.1}$$

where the matrix-function $B(u, v)$ is defined and continuous in the region

$$-\infty < u < \infty, \qquad t_0 \leqslant v \leqslant \infty. \tag{2.2}$$

Furthermore, $B(u, v)$ is to be periodic in u, so that for some $\omega > 0$ we have

$$B(u + \omega, v) = B(u, v). \tag{2.3}$$

In addition, it is to be of bounded variation in v, in the following restrictive sense; it is to be absolutely continuous in v, and for some measurable function $\eta(t) \geqslant 0$, with

$$\int_{t_0}^{\infty} \eta(t)\, dt < \infty, \tag{2.4}$$

we must have almost everywhere

$$\left| \frac{\partial B(u, v)}{\partial v} \right| \leqslant \eta(v). \tag{2.5}$$

Denoting by n the order of the matrices $A(t)$, $B(u, v)$, we denote by

$$\mu_1(v), \ldots, \mu_n(v), \qquad t_0 \leqslant v \leqslant \infty, \tag{2.6}$$

the characteristic exponents of the system, with periodic coefficients,

$$y' = B(t, v)\, y. \tag{2.7}$$

We wish to apply a result for (1.2) which requires the distinctness of the characteristic roots of A_0. Accordingly, we will assume that the characteristic exponents (2.6) are distinct when $v = \infty$. If t_0 is taken sufficiently large, we may then suppose them distinct when $t_0 \leqslant v$, and as continuous functions of v in $t_0 \leqslant v \leqslant \infty$; we return in more detail to the topic of dependence of (2.7) on v in the next section. We are interested in circumstances under which (1.1) has a set of n solutions of the form

$$y^{(r)}(t) = c_r(t) \left\{ \exp \int_{t_0}^{t} \mu_r(v)\, dv \right\} (1 + o(1)), \qquad r = 1, \ldots, n, \tag{2.8}$$

for various periodic, nonzero vectors $c_r(t)$.

For stable cases of (1.3), we have then

THEOREM 1. *In addition to the above assumptions, let all the* $\mu_r(v)$ *have zero real part,* $t_0 \leqslant v \leqslant \infty$. *Then* (1.1) *has a set of solutions of the form* (2.8).

Levinson's result for (1.2) [2, 5, 6] provides an assortment of conditions on the characteristic roots of $A_0 + B(t)$ under which asymptotic integration is possible. The other extreme, from Theorem 1, is for example given by

THEOREM 2. *Let all the* (2.6) *have distinct real parts for* $v = \infty$, *(and so for large* v*)*; *then the asymptotic integration* (2.8) *holds.*

3. THE FLOQUET THEORY WITH PARAMETER

We start as usual with the fundamental matrix solution of (2.7), the matrix $X(t, v)$ given by

$$\frac{\partial X(t, v)}{\partial t} = B(t, v)\, X(t, v), \qquad X(0, v) = I, \tag{3.1}$$

where I is the unit matrix. Since $X(t, v)$ is nonsingular, we may define

$$D(v) = \omega^{-1} \log X(\omega, v); \tag{3.2}$$

we give later a definition which makes $D(v)$ a continuous function of v, not of course unique, but pass over this point for the moment. Following the course of the Floquet theory, we then observe that

$$X(t, v)\, e^{\omega D(v)}, \qquad X(t + \omega, v)$$

must coincide, since they coincide when $t = 0$ and also satisfy the same differential equation in t; thus

$$P(t, v) = X(t, v)\, e^{-tD(v)} \tag{3.3}$$

is periodic in t, with period ω, and is nonsingular. We have the identity

$$X(t, v) = P(t, v)\, e^{tD(v)}. \tag{3.4}$$

We now consider dependence on v of these entities, for t subject to $0 \leqslant t \leqslant \omega$. We have first that $X(t, v)$ is bounded. By the Gronwall-Bellman inequality we have from (3.1) that

$$| X(t, v) | \leqslant | I | \exp \int_0^\omega | B(t, v) | \, dt, \tag{3.5}$$

and a similar argument shows that $\{X(t, v)\}^{-1}$ is bounded.

Next we must consider $\partial X(t, v)/\partial v$. From the differentiated form of (3.1),

$$\frac{\partial^2 X}{\partial t}\,\partial v = B\frac{\partial X}{\partial v} + \frac{\partial B}{\partial v}X, \qquad \frac{\partial X(0, v)}{\partial v} = 0,$$

the fundamental inequality gives

$$\left|\frac{\partial X}{\partial v}\right| \leqslant \sup_{0 \leqslant t \leqslant \omega}\left\{\left|\frac{\partial B(t, v)}{\partial v}\right| \mid X(t, v) \mid\right\} \exp \int_0^\omega \mid B(t, v) \mid dt,$$

whence we derive

$$\left|\frac{\partial X(t, v)}{\partial v}\right| = O(\eta(v)), \tag{3.6}$$

uniformly for $0 \leqslant t \leqslant \omega$, $v \geqslant t_0$.

We can now consider the variation of $D(v)$. We make (3.2) more precise by specifying that

$$D(v) = (2\pi i\omega)^{-1}\int_L (\log\lambda)\,(\lambda I - X(\omega, v))^{-1}\,d\lambda. \tag{3.7}$$

Here the contour L must encircle each eigenvalue of $X(\omega, v)$ once positively, but must not encircle the origin; we may give $\log \lambda$ any determination on L, subject to continuity. We choose L to satisfy these requirements in respect of $X(\omega, \infty)$. Since the eigenvalues of the latter are the limits of those of $X(\omega, v)$, as $v \to \infty$, the contour L will then satisfy our requirements in respect of $X(\omega, v)$ for $v \geqslant t_0$, if we suppose t_0 taken sufficiently large.

We may then differentiate (3.7) under the integral sign with respect to v, to obtain the result that

$$D'(v) = O(\eta(v)), \tag{3.9}$$

in view of (3.6). Of course, (3.7) shows immediately that $D(v)$ is bounded, and a continuous function of v. In a similar way, we have that $\exp(\pm\, tD(v))$ are bounded for $0 \leqslant t \leqslant \omega$. As their differentiability, we have

$$\exp(-\,tD(v)) = (2\pi i)^{-1}\int_L \exp(-\,t\omega^{-1}\log\lambda)\,(\lambda I - X(\omega, v))^{-1}\,d\lambda,$$

and hence

$$\frac{\partial}{\partial v}\exp(-\,tD(v)) = O(\eta(v)), \tag{3.10}$$

still for $0 \leqslant t \leqslant \omega$.

Finally we must consider the dependence of $P(t, v)$ on v. It follows from (3.3) that $P(t, v)$ is bounded, uniformly for $t_0 \leqslant v$ and $0 \leqslant t \leqslant \omega$. Since however $P(t, v)$ is periodic in t, the last requirement may be dropped. Likewise, we have that $\{P(t, v)\}^{-1}$ is bounded. As to differentiability, we note that, by (3.3),

$$\frac{\partial P(t, v)}{\partial v} = \frac{\partial X(t, v)}{\partial v} e^{tD(v)} + X(t, v) \frac{\partial}{\partial v} e^{tD(v)}$$

and so, by (3.6), (3.10),

$$\frac{\partial P(t, v)}{\partial v} = O(\eta(v)), \tag{3.11}$$

for $0 \leqslant t \leqslant \omega$, and so for all t by periodicity.

Finally, we recall the differential equation satisfied by $P(t, v)$ in t, on the basis that (3.4) satisfies (3.1). Substituting (3.4) in (3.1), we get, denoting $\partial P(t, v)/\partial t$ by $P_1(t, v)$,

$$P_1(t, v)\, e^{tD(v)} + P(t, v)\, D(v)\, e^{tD(v)} = B(t, v)\, P(t, v)\, e^{tD(v)},$$

whence we have

$$P_1(t, v) + P(t, v)\, D(v) = B(t, v)\, P(t, v). \tag{3.12}$$

4. Reduction to the Case (1.2)

We now take the equation (1.1), with the form (2.1). For a solution $y(t)$ of (1.1) we define $z(t)$ by

$$y(t) = P(t, t)\, z(t), \tag{4.1}$$

and calculate the differential equation satisfied by $z(t)$. We write $P_2(t, v)$ for $\partial P(t, v)/\partial v$. Differentiation of (4.1) gives

$$y'(t) = P_1(t, t)\, z(t) + P_2(t, t)\, z(t) + P(t, t)\, z'(t), \tag{4.2}$$

while (1.1) gives

$$y'(t) = A(t)\, y(t) = B(t, t)\, P(t, t)\, z(t). \tag{4.3}$$

In addition, we have from (3.12) that

$$P_1(t, t) + P(t, t)\, D(t) = B(t, t)\, P(t, t).$$

Taken together with (4.2-3), we deduce that

$$P(t, t)\, D(t)\, z(t) = P_2(t, t)\, z(t) + P(t, t)\, z'(t),$$

and so

$$z'(t) = \{D(t) - (P(t, t))^{-1} P_2(t, t)\} z(t). \tag{4.4}$$

We thus reach the form (1.2), with

$$A_0 = D(\infty), \qquad B(t) = D(t) - D(\infty), \qquad C(t) = (P(t, t))^{-1} P_2(t, t).$$

In justification of these statements, we have that $B(t)$ so given tends to zero as $t \to \infty$, and is of bounded variation over (t_0, ∞), by (3.9), (2.4), while $C(t)$ so given is absolutely integrable over (t_0, ∞) by (3.11), and the boundedness of $\{P(t, t)\}^{-1}$.

The theorems stated now follow from Levinson's result, since the characteristic exponents (2.6) are none other than the eigenvalues of $D(v)$.

5. Application to the Stable Second-Order Case

Consider now the equations

$$u'' + (1 + h(t)\, \varphi(t)) = 0, \tag{5.1}$$

$$u'' + (1 + \epsilon\varphi(t))\, u = 0, \tag{5.2}$$

where $h(t)$, $\varphi(t)$ are continuous for $t \geqslant t_0$, and $\varphi(t)$ is of period $\omega > 0$. For (5.2), with $\epsilon = 0$, the characteristic multipliers will be $\exp(\pm i\omega)$, if (5.2) is considered as a linear system with coefficients of period ω. Provided that ω is not a multiple of π, these multipliers will not be equal to ± 1, and will be distinct, lying on the unit circle. Thus, for small ϵ, real, the characteristic multipliers will be complex conjugate, distinct, and on the unit circle. Furthermore, the characteristic exponents will be purely imaginary, and distinct, and expressible as power series

$$\pm \mu(\epsilon) = \pm i \sum_{r=0}^{\infty} \gamma_r \epsilon^r, \tag{5.3}$$

convergent for small ϵ, where the y_r are real.

Let now $h(t)$ in (5.1) be real-valued, absolutely continuous, and of bounded variation over (t_0, ∞), and furthermore tend to zero as $t \to \infty$, though this is not essential. We may then conclude that (5.1) has solutions of the form

$$(1 + o(1)) \exp \left\{\pm i \sum_{r=0}^{\infty} \gamma_r \int_{t_0}^{t} \{h(t)\}^r \, dt\right\}, \tag{5.4}$$

if t_0 is taken sufficiently large. The case $\varphi(t) = \cos kt$ is considered by iterative procedures by Kelman and Madsen (10), in extension of earlier work (8, 9). The iteration has to be taken until we reach a power of $h(t)$ which is absolutely integrable at ∞, if such a power exists. The same iterative procedures can of course be applied in the case of a small parameter ϵ in place of $h(t)$—so that (5.2) is essentially the Mathieu equation—to show that the coefficients γ_r are the same as those arising in the expansion of the characteristic exponents, as in the cases (5.3), (5.4).

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